1-4 DIMENSIONAL ANALYSIS

Dimensions

Experience has shown that there are three basic ways to describe any physical quantity: the space it takes up, the matter it contains, and how long it persists. All descriptions of matter, relationships, and events are combinations of these three basic characteristics. All measurements can be reduced ultimately to the measurement of length, time, and mass. Any physical quantity, no matter how complex, can be expressed as an algebraic combination of these three basic quantities. Speed, for example, is a length per time.

Length, time, and mass therefore have a significance far beyond that of providing the basis of a system of units. They specify the three **primary dimensions**. We use the abbreviations $[L]$, $[T]$, and $[M]$ for these primary dimensions. The **dimension** of a physical quantity is the algebraic combination of $[L]$, $[T]$, and $[M]$ from which the quantity is formed. The speed v provides an example. The dimension of v is

$$[v] = [L/T] \quad \text{or} \quad [LT^{-1}].$$

Do not confuse the dimension of a quantity with the units in which it is measured. A speed may have units of meters per second, miles per hour, or, for that matter, light-years per century. All of these different choices of units are consistent with the dimension $[LT^{-1}]$. In what follows, the square brackets, as used here, indicate that we are dealing with dimensions.

Any physical quantity has dimensions that are algebraic combinations $[L^q T^r M^s]$ of the primary dimensions, where the superscripts q, r, and s refer to the order (or power) of the dimension. Thus, for example, an area has dimension $[L^2]$. If all of the exponents q, r, and s are zero, the combination will be dimensionless. The exponents q, r, and s can be positive integers, negative integers, or even fractional powers.

Dimensional Analysis

Study of the dimensions of an equation—*dimensional analysis*—is an important exercise with several different uses in physics. Any equation that relates physical quantities must have consistent dimensions; that is, *the dimensions on one side of an equation must be the same as those on the other side.* This provides a valuable check for any calculation. Dimensional analysis can also reveal *scaling laws* (see Section 1–7), which describe how changing one quantity in a physical situation requires changes in others. Finally, when there is reason to believe that only certain physical quantities can enter into a physical situation, dimensional analysis can provide us with powerful insights.

Let's look at some examples of dimensional analysis. In Chapter 7, we derive a relation between the height h of a dropped object and the speed of that object. This relation involves the *acceleration of gravity*, g, a quantity whose dimension is $[g] = [LT^2]$. The relation reads

$$gh = \frac{1}{2}v^2.$$

Let's compare the dimensions on each side of this equation. The dimension of h is $[L]$, so the left-hand side has dimensions $[LT^{-2}][L] = [L^2T^{-2}]$. The right-hand side has the dimensions of speed squared, $[LT^{-1}]^2 = [L^2T^{-2}]$. Thus the dimensions match. If, through error, we had written a relation $gh^2 = \frac{1}{2}v^2$, then this check would have revealed the error. Note that dimensional analysis does not help us understand the numerical factor $\frac{1}{2}$.

EXAMPLE 1–2 Newton's law of universal gravitation gives the force between two objects of mass, m_1 and m_2, separated by a distance r, as

$$F = G\left(\frac{m_1m_2}{r^2}\right).$$

Use dimensional analysis to find the units of the gravitational constant, G.

Solution: First, the dimensions of the two sides of the equation must match. In the previous section, we learned that the unit of force is the newton, equivalent to $\text{kg}\cdot\text{m/s}^2$. Using these units, the dimensions of force must be $[MLT^{-2}]$. We now know the dimensions of every quantity in the equation for gravitational force except G. Writing the dimensions for both sides gives

$$[MLT^{-2}] = [G][M][M][L^2] = [G][M^2L^{-2}].$$

Note that the individual dimensions can be consolidated inside the square brackets or left within their own brackets—whichever is easiest. We solve for the dimension of G as

$$[G] = \frac{[MLT^{-2}]}{[M^2L^{-2}]} = [MLT^{-2}][M^{-2}L^2] = [M^{-1}L^3T^{-2}].$$